Non-Linear Bayesian Orbit Determination Based on the Generalized Admissible Region

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Abstract—**In this paper, we propose a non-linear Bayesian estimation technique where, for a set of observations, the physical limits of the knowledge of the observed object are represented not as likelihood functions but as probability density functions (pdfs). When the codimension of the observations are high, a direct numerical implementation of Bayes' theorem is practical. The pdfs are mapped analytically in time by means of a special solution to the Fokker-Planck equations for deterministic systems. This approach requires no a priori information, enables direct comparison of observations with any probabilistic data, and is robust to outlier observations.**

I. INTRODUCTION

Situational awareness of resident space objects (RSOs) such as active satellites and space debris is known to be a *data starved* problem compared to traditional estimation problems in that objects may not be observed for days if not weeks [1]. Therefore, consistent characterization of the uncertainty associated with each state estimate is crucial in maintaining an accurate catalog of RSOs. Recently in astrodynamics, much attention has been given to the non-linear deformation of uncertainty for the orbiter problem as well as its applications to object correlation, observation association, and conjunction assessment [2]–[5]. Simultaneously, the motion of satellites in Earth orbit is *well-modeled* in that it is particularly amenable to having their solution and their uncertainty described through analytic or semi-analytic techniques. Even when stronger non-gravitational perturbations such as solar radiation pressure and atmospheric drag are encountered, these perturbations generally have deterministic components that are substantially larger than their time-varying stochastic components [6], [7].

Traditionally, orbit determination has been conducted with some type of batch or sequential estimation algorithm, whose *a priori* information is supplied via geometric techniques [8], [9]. Although conventional initial orbit determination (IOD) works well for celestial bodies that are predominantly influenced by gravity and can be observed over many nights, it is less effective for RSOs which are observed in short bursts and experience many perturbing forces including atmospheric drag, irregularities of the central body, and solar radiation pressure, just to name a few. Moreover, since IOD is geometry-based, it assumes the association of observations (i.e. that they were of the same object) and does not provide error bounds to its state estimates. Especially in the realm of optical (bearing-only) observations, these difficulties are referred to as *too short arc* (TSA) [10]. The more general problem of multiple target tracking using bearing only sensors has also been tackled outside of astrodynamics, but most solutions similarly require a reference state, a Gaussian assumption on the error distribution, or great computational power [11], [12].

In this paper, we propose a non-linear Bayesian (initial) orbit determination technique where the observations are expressed not as likelihood functions but rather as pdfs in the state space. The integral of the likelihood over the entire state space is divergent for underdetermined systems, but by placing a few physical constraints, one can limit the domain in which the truth state lies in, giving us tractable compact pdfs. Consequently, these pdfs represent not only the uncertainty of an observation due to errors but also the physical limits of the knowledge of variables that are not directly observed: range and range-rate for bearing-only observations, for example. In the TSA problem, the domain of the pdf is called the *admissible region* [13], [14]. Furthermore, if the pdfs are of high codimension, as is the case for bearing-only observations considering angular rates as directly observed variables, then they can be combined efficiently via a numerical evaluation of Bayes' theorem. The above approach is attractive because it requires no *a priori* information, enables direct comparison with any probabilistic data at all stages of the estimation, and is robust to outlier observations. The association of observations and state estimation are conducted simultaneously for any number of observations or dynamical model. It also handles well ambiguities in the number of orbit periods between observations. The pdfs are mapped analytically in time by means of a special solution to the Fokker-Planck equations for deterministic systems [15], [16]. We use state transition tensors (STTs) to approximately describe the solution flow to the dynamics, as STTs are defined even for systems with no closed-form solutions [15], [17].

The outline of this paper is as follows. We first introduce the necessary mathematical concepts, including the generalized admissible region, the combination of pdfs via Bayes' rule, and the analytical propagation of pdfs (*Background*). Next, an example is shown for processing optical observations of two Earth-orbiting objects in the same orbit but of different phase (*Example*). Our theory provides a semi-analytical framework that is distinct from other approaches currently being developed in the field for dealing with the highly non-linear dynamical environment that RSOs encounter.

II. Background

In this section, we only aim to give a brief introduction on the relevant mathematical results. Further discussions on the implementation of the theory discussed in this section can be found in the authors' previous papers [5], [18], [19].

A. The Generalized Admissible Region

The batch and sequential filters, the two most commonly utilized estimation techniques in orbit determination, can be derived as a linear, unbiased, maximum *a posteriori* estimator, where the observation and *a priori* errors are taken into account up to their second moments.

Theorem 2.1: Suppose observation error is unbiased with a covariance of R and the *a priori* state uncertainty is similarly unbiased with a covariance of $\bar{P}$. Then, the linear, unbiased, maximum *a posteriori* estimate $\hat{\mathbf{x}}$ and its associated covariance P is

$$P = \left[\tilde{H}^{\mathrm{T}} R^{-1} \tilde{H} + \bar{P}^{-1}\right]^{-1} \tag{1}$$

$$\hat{\mathbf{x}} = P(\tilde{H}^{\mathrm{T}} R^{-1} \mathbf{y} + \bar{P}^{-1}\bar{\mathbf{x}}), \tag{2}$$

where $\mathbf{y}$ is the observation deviation from the reference and $\tilde{H}$ is the linearized relationship between the observations and the states.

Proof: See [8]. ■

Furthermore, dynamical modeling errors can be included in the analysis as white noise (state noise compensation) or a Gauss-Markov process (dynamic model compensation) [8]. A main limitation of this approach is that the filtering and estimation process is built around an assumption that the true solution is close to the mean solution found from the pdf. For many problems that involve uncorrelated observations with large uncertainties, however, this is a poor starting point. Indeed, the application of particle filters and sigma point filters, for example [20], [21], have been developed to deal with these limitations.

The method proposed in this paper, on the other hand, takes a fundamental approach to the problem, with our novelty being that we define and describe a fully analytical approach to the general orbit determination problem. We start by describing any type of observation as a probability density function (pdf) of the state of the observed object.

Definition 2.1: Let $\mathcal{X}$ be the n-dimensional state space and define an invertible transformation $\mathcal{F} : \mathcal{X} \mapsto \mathcal{Y}$ to some space $\mathcal{Y}$. Further, let $\bar{\mathcal{Y}}$ be a m $(\leq n)$ dimensional subset of $\mathcal{Y}$. An *attributable vector* $\mathfrak{Y}^{t^0} \equiv \mathfrak{Y}^0 \in \bar{\mathcal{Y}}$ is a vector containing all of the directly observed variables for a given observation at time t^0.

Definition 2.2: Any variables in set $\mathcal{Y} \setminus \bar{\mathcal{Y}}$ are referred to as *unobserved*.

Definition 2.3: Suppose that, given some set of criteria $\mathcal{C}$, A is a compact set in $\mathcal{X}$ that meet $\mathcal{C}$. Then, the *generalized admissible region* $F_{\mathcal{C}}[\mathbf{X}(t^0); \mathfrak{Y}^0]$ is a pdf over $\mathcal{X}$ assigned to an attributable vector $\mathfrak{Y}^0$ such that the probability p that the observed object exists in region $B \subset A$ at time t^0 is

$$p[\mathbf{X}(t^0)] = \int_B F_{\mathcal{C}}[\mathbf{X}(t^0); \mathfrak{Y}^0] \mathrm{d}X_1^0 \mathrm{d}X_2^0 \ldots \mathrm{d}X_n^0, \tag{3}$$

where $\mathbf{X}(t^0) \in \mathcal{X}$ and

$$\mathbf{X}(t^0) \equiv \mathbf{X}^0 = (X_1^0, X_2^0, \ldots, X_n^0). \tag{4}$$

Note that we impose $\int_A F_{\mathcal{C}}[\mathbf{X}(t^0); \mathfrak{Y}^0] \mathrm{d}\mathbf{X}^0 = 1$.

Conceptually, the generalized admissible region expresses our *limited* knowledge regarding the unobserved variables. In conventional filtering, pdfs of the observations only describe the error in the attributable vector and are realized in the state space as likelihoods. For underdetermined systems, the integral of the likelihood function over the state space is divergent as we gain no information from the observations in coordinate directions corresponding to the unobserved variables. We realize, however, that knowledge in these directions is not completely *lacking* for many real-world systems as the likelihood function may suggest. That is, we may add physical constraints $\mathcal{C}$ to the observed object's state such that we define a compact pdf F still representative of all relevant states. Furthermore, the uncertainties due to errors in the attributable vector are often much smaller than the uncertainties due to our limited knowledge in the unobserved variables, motivating the following remark.

Remark 2.1: The generalized admissible region can be regarded as a compact $n - m$ dimensional manifold embedded in an n dimensional space.

B. Combination of Observations

By converting observation information into a pdf and propagating them to a common epoch, observations can be rationally combined with any other probabilistic data using Bayes' theorem, be it new observations or density information from the two-line element (TLE) catalog, for example.

Definition 2.4: An event $E_{\mathcal{O}^1}(V^\tau)$ where some observation $\mathcal{O}^1$ made at time t^1 is *consistent* with some region V^τ in state space at time τ if, given P is a probability measure and $f[\mathbf{X}(t^1); \mathcal{O}^1]$ is the pdf associated with $\mathcal{O}^1$,

$$P[E_{\mathcal{O}^1}(V^\tau)] = \int_{V^\tau} \left\{\mathcal{T}(\tau, t^1) \circ f[\mathbf{X}(t^1); \mathcal{O}^1]\right\} \mathrm{d}\mathbf{X}, \tag{5}$$

where $\mathcal{T}(\tau, t^1)$ is a transformation that maps f from time t^1 to τ and $\mathbf{X} \equiv \mathbf{X}(\tau)$. Here, the term *observation* is used in the statistical sense: it is not limited to physical observations but rather apply to results from any kind of experiment or analysis. We discuss transformation $\mathcal{T}$ in more detail in the following subsection.

Corollary 2.1: Let $E_{\mathcal{O}^2=\mathcal{O}^1}$ be an event where some observation $\mathcal{O}^2$ made at time t^2 is of the same object as $\mathcal{O}^1$. Then, if the pdf associated with $\mathcal{O}^2$ is $g[\mathbf{X}(t^2);\mathcal{O}^2]$,

$$P[E_{\mathcal{O}^1}(V^\tau)|E_{\mathcal{O}^2=\mathcal{O}^1}] = \int_{V^\tau} h[\mathbf{X}(\tau)]\mathrm{d}\mathbf{X} \tag{6}$$

where

$$h[\mathbf{X}(\tau)] = \frac{\{\mathcal{T}(\tau,t_1)\circ f[\mathbf{X}(t^1);\mathcal{O}^1]\}\{\mathcal{T}(\tau,t_2)\circ g[\mathbf{X}(t^2);\mathcal{O}^2]\}}{\int\{\mathcal{T}(\tau,t_1)\circ f[\mathbf{X}(t^1);\mathcal{O}^1]\}\{\mathcal{T}(\tau,t_2)\circ g[\mathbf{X}(t^2);\mathcal{O}^2]\}\mathrm{d}\mathbf{X}}, \tag{7}$$

and $|$ indicates conditional probability. The integral in (7) is over the entire state space.

Proof: Given $E_{\mathcal{O}^1}(V^\tau)$ true, for $E_{\mathcal{O}^2=\mathcal{O}^1}$ to also be true, we require $E_{\mathcal{O}^2}(V^\tau)$ for any choice of V^τ. Therefore,

$$\begin{aligned} P[E_{\mathcal{O}^2=\mathcal{O}^1}|E_{\mathcal{O}^1}(V^\tau)] &= P[E_{\mathcal{O}^2}(V^\tau)] \\ &= \int_{V^\tau}\{\mathcal{T}(\tau,t^2)\circ g[\mathbf{X}(t^2);\mathcal{O}^2]\}\mathrm{d}\mathbf{X}. \end{aligned} \tag{8}$$

Then apply Bayes' theorem [22]. ■

Now, for systems where the pdfs f and g are of high codimension, the sparseness of the problem allows for efficient ways to associate observations and simultaneously obtain a state estimate.

Corollary 2.2: For observations $\mathcal{O}^1$ and $\mathcal{O}^2$ such that

$$\dim\left\{f[\mathbf{X}(t^1);\mathcal{O}^1]\right\} + \dim\left\{g[\mathbf{X}(t^2);\mathcal{O}^2]\right\} < \dim(\mathbf{X}), \tag{9}$$

$P[E_{\mathcal{O}^2=\mathcal{O}^1}] = 0$ if for all $\mathbf{X}$,

$$\{\mathcal{T}(\tau,t^1)\circ f[\mathbf{X}(t^1);\mathcal{O}^1]\}\{\mathcal{T}(\tau,t^2)\circ g[\mathbf{X}(t^2);\mathcal{O}^2]\} = 0. \tag{10}$$

Proof: Apply the theory of general position to Remark 2.1 [23]. ■

Therefore, for a set of observations that satisfy (9), their association does not have to be assumed as it is determined as a consequence of the combination process (6). We conclude that this method can be particularly robust to outlier observations.

Furthermore, if f and g have high codimension, we can employ linear extrapolation to rapidly discretize the pdfs and evaluate (6) numerically without having to face the consequences of the so called *curse of dimensionality* [18]. (9) suggests that if the two pdfs are associated with the same object, then more and more constraints must be met for their intersection to have higher dimensions [19]. The *a posteriori* pdfs, then, would most likely span a very small region in state space if not singularly defined at a point. Subsequent application of (9) becomes computationally trivial. An example of the above would be bearing-only observations where angular rates are considered as directly observed: if the state space is 6-dimensional, the codimension of each pdf is 4. Refer to Section III for a numerical implementation.

C. Analytical Non-Linear Propagation of Uncertainty

In general, to find a transformation $\mathcal{T}$ that maps a pdf in time as discussed in Definition 2.4, one must solve the Fokker-Planck differential equations.

Theorem 2.2: For a dynamical system $\dot{\mathbf{X}} = \boldsymbol{f}(\mathbf{X},t)$ that satisfies the Itô stochastic differential equation, the time evolution of a probability density function (pdf) $p(\mathbf{X},t)$ over $\mathbf{X}$ at time t is described by the Fokker-Planck equation [24]

$$\begin{aligned} \frac{\partial p(\mathbf{X},t)}{\partial t} &= -\frac{\partial}{\partial \mathbf{X}_i}\{p(\mathbf{X},t)\boldsymbol{f}_i(\mathbf{X},t)\} \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial \mathbf{X}_i \partial \mathbf{X}_j}\left[p(\mathbf{X},t)\left\{G(\mathbf{X},t)Q(t)G^{\mathrm{T}}(\mathbf{X},t)\right\}_{ij}\right], \end{aligned} \tag{11}$$

where a single subscript indicates vector components and a double subscript indicates matrix components. Einstein notation is assumed hereafter. Matrices G and Q characterize the diffusion.

Proof: See [24], [25]. ■

However, for deterministic systems, a special solution exists.

Lemma 2.1: For a deterministic dynamical system, the probability $\Pr(\mathbf{X}\in\mathcal{B}) = \int_{\mathcal{B}} p(\mathbf{X},t)\mathrm{d}\mathbf{X}$ over some volume $\mathcal{B}$ in phase space is an integral invariant.

Proof: Canceling out terms in (11) pertaining to diffusion, we have

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = -\frac{\partial}{\partial \mathbf{X}_i}\{p(\mathbf{X},t)\boldsymbol{f}_i(\mathbf{X},t)\}. \tag{12}$$

Thus, $\Pr(\mathbf{X}\in\mathcal{B})$ satisfies the sufficient condition for integral invariance. See [15], [26], [27] for full proof. ■

Theorem 2.3: For a deterministic dynamical system, given solution flow $\boldsymbol{\phi}(t;\mathbf{X}^0,t^0)$ to the dynamics for initial conditions $\mathbf{X}(t^0)=\mathbf{X}^0$, the pdf $p(\mathbf{X},t)$ is expressed as

$$p(\mathbf{X},t) = p[\boldsymbol{\phi}(t;\mathbf{X}^0,t^0),t] = p(\mathbf{X}^0,t^0)\left|\frac{\partial \mathbf{X}}{\partial \mathbf{X}^0}\right|^{-1}, \tag{13}$$

where $|\cdot|$ indicates the determinant operator.

Proof: From the substitution rule and integral invariance, for some phase volume $\mathcal{B}$,

$$\begin{aligned} \Pr(\mathbf{X}\in\mathcal{B}) &= \int_{\mathcal{B}} p(\mathbf{X},t)\mathrm{d}\mathbf{X} && (14) \\ &= \int_{\mathcal{B}^0} p(\mathbf{X},t)\left|\frac{\partial \mathbf{X}}{\partial \mathbf{X}^0}\right|\mathrm{d}\mathbf{X}^0 && (15) \\ &= \int_{\mathcal{B}^0} p(\mathbf{X}^0,t^0)\mathrm{d}\mathbf{X}^0, && (16) \end{aligned}$$

where $\mathcal{B}^0$ is the phase volume corresponding to $\mathcal{B}$ at time t^0. Equating integrands in (15) and (16), we find (13). See [15], [16] for full proof. ■

Therefore, a pdf can be expressed analytically for all time given an analytical expression of both the initial pdf and the solution flow. For systems with no closed-form solution flow, we may obtain an approximate analytical solution from a Taylor series expansion.

Definition 2.5: If a Taylor series expansion of a dynamical differential equation $\dot{\mathbf{X}} = \boldsymbol{f}(\mathbf{X}, t)$ is taken up to order m about some reference trajectory $\mathbf{X}^*$, the time derivative of the state deviation $\dot{\mathbf{x}} = \dot{\mathbf{X}} - \dot{\mathbf{X}}^*$ is expressed as

$$\dot{\mathbf{x}}_i(t) = \sum_{p=1}^{m} \frac{1}{p!} A_{i,k_1 \dots k_p} \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p}, \tag{17}$$

where the subscripts indicate the component of each tensor and A is the *local dynamics tensor* (LDT) of order p [15]

$$A_{i,k_1 \dots k_p} = \left. \frac{\partial^p \boldsymbol{f}}{\partial \mathbf{X}^0_{k_1} \dots \partial \mathbf{X}^0_{k_p}} \right|_* . \tag{18}$$

The comma in the subscript simply denotes that the i-th component is not summed over. The subscript $*$ indicates that A is evaluated over the reference trajectory $\mathbf{X}^*$.

Definition 2.6: If a Taylor series expansion of a solution flow $\mathbf{X}(t; \mathbf{X}^0, t^0) = \boldsymbol{\phi}(t; \mathbf{X}^0, t^0)$ for a dynamical system $\dot{\mathbf{X}} = \boldsymbol{f}(\mathbf{X}, t)$ is taken up to order m about some reference trajectory $\mathbf{X}^*$, $\mathbf{x}$ is expressed as

$$\mathbf{x}_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1 \dots k_p} \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p}, \tag{19}$$

where Φ is the *state transition tensor* (STT) of order p [15]

$$\Phi_{i,k_1 \dots k_p} = \left. \frac{\partial^p \mathbf{X}_i}{\partial \mathbf{X}^0_{k_1} \dots \partial \mathbf{X}^0_{k_p}} \right|_* . \tag{20}$$

Corollary 2.3: The time derivative of the STT of order p is a function only of the LDT and STT of order up to p.

Proof: Take the time derivative of (19) and equate to (17). Details of this differential equation is in [5].

Corollary 2.4: The mean $\mathbf{M}(t) = \mathbf{m}(t) + \mathbf{X}^*(t)$ and covariance matrix $[P](t)$ of a pdf $p[\mathbf{X}(t)]$ can be propagated non-linearly as

$$\mathbf{m}_i(t) = \sum_{p=1}^{m} \frac{1}{p!} \Phi_{i,k_1 \dots k_p} \times \int_\infty p(\mathbf{x}^0) \left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right|^{-1} \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p} \mathrm{d}\mathbf{x}^0 \tag{21}$$

$$[P]_{ij}(t) = \left[\sum_{p=1}^{m} \sum_{q=1}^{m} \frac{1}{p!q!} \Phi_{i,k_1 \dots k_p} \Phi_{j,l_1 \dots l_q} \times \int_\infty p(\mathbf{x}^0) \left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right|^{-1} \mathbf{x}^0_{k_1} \dots \mathbf{x}^0_{k_p} \mathbf{x}^0_{l_1} \dots \mathbf{x}^0_{l_q} \mathrm{d}\mathbf{x}^0 \right] - \mathbf{m}_i(t) \mathbf{m}_j(t) \tag{22}$$

Proof: From (13),

$$\begin{aligned} \mathbf{M}(t) &= \int_\infty \mathbf{X}(t) p[\mathbf{X}(t)] \mathrm{d}\mathbf{X} \\ &= \int_\infty \boldsymbol{\phi}(t; \mathbf{X}^0, t^0) p(\mathbf{X}^0) \left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right|^{-1} \mathrm{d}\mathbf{X}^0 \end{aligned} \tag{23}$$

$$\begin{aligned} [P](t) &= \int_\infty [\mathbf{X}(t) - \mathbf{M}(t)]^{\mathrm{T}} [\mathbf{X}(t) - \mathbf{M}(t)] p[\mathbf{X}(t)] \mathrm{d}\mathbf{X} \\ &= \int_\infty [\boldsymbol{\phi}(t; \mathbf{X}^0, t^0) - \mathbf{M}(t)]^{\mathrm{T}} \times \\ &\qquad [\boldsymbol{\phi}(t; \mathbf{X}^0, t^0) - \mathbf{M}(t)] p(\mathbf{X}^0) \left| \frac{\partial \mathbf{X}}{\partial \mathbf{X}^0} \right|^{-1} \mathrm{d}\mathbf{X}^0 \end{aligned} \tag{24}$$

Then, subtract the reference trajectory and substitute (19). ■

III. Example

In this section, we discuss association and initial orbit determination results for optical observation simulations of two objects in the same circular MEO orbit but of different phase. The purpose of this example is to graphically highlight how our method works without the need of *a priori* information regarding the state of the observed objects, assumptions on the association of observations, or specification of the dynamical system used. The Keplerian orbital elements of the first object (*Object 1*) is:

$$(a, e, i, \Omega, \omega, M) = \tag{25}$$

$$(3.9994, 0.0006, 1.1284, 4.9148, 4.2128, 2.9461), \tag{26}$$

where units are in Earth radii and rad. The second object (*Object 2*) proceeds the first in mean anomaly by $2\pi/3$ rad.

We simulated ground-based zero-error observations of right ascension (α), declination (δ), and their time derivatives. Thus $\mathfrak{X} = (\alpha, \delta, \dot{\alpha}, \dot{\delta}, \mathrm{h}, \Theta, \phi)$, where (Θ, ϕ) is the angular position and h is the altitude of the observation point. We then generated generalized admissible regions $F[\mathbf{X}(t^0); \mathfrak{X}^0]$ for each observation constrained by orbit energy, apoapsis and periapsis height, and limits on range / range-rate; refer to Fujimoto and Scheeres for details [18]. The error-free approximation is good since the state uncertainty due to observation error is much less than that due to lack of observability in range and range-rate [28]. It also simplifies computation as, from Remark 2.1, the pdfs are 2-dimensional manifolds embedded in a 6-dimensional space, making the problem sparse. We assumed no *a priori* information regarding the observed objects, and thus used a uniform initial pdf in the topocentric spherical coordinates.

We chose the Poincaré orbital element space $\mathbf{X} = (\mathfrak{L}, \mathfrak{l}, \mathfrak{G}, \mathfrak{g}, \mathfrak{H}, \mathfrak{h})$ as our state space variables since they are not only non-singular but also canonical: they can thus be naturally grouped by their coordinate-momenta symplectic pairs [9], [28]. The non-singular property assures that the pdf mapping function $\mathcal{T}(\tau, t_0)$ is well defined everywhere. Refer to Appendix A for the definition of Poincaré variables. For computation, the Poincaré space was discretized such that:

$$\mathbf{X}_{\min} = (4.4621, 0, -3, -3, -4, -4)^{\mathrm{T}} \tag{27}$$

$$\mathbf{X}_{\max} = (12.6206, 6.2832, 3, 3, 4, 4)^{\mathrm{T}}, \tag{28}$$

where $\mathbf{X}_{\min}$ and $\mathbf{X}_{\max}$ are the lower and upper limits of the discretization, respectively. Units are in Earth radii, hours, and radians. In each coordinate direction, the space was divided into 100 ($\mathfrak{L}$,) 77 ($\mathfrak{l}$,) 73 ($\mathfrak{G}$,) 73 ($\mathfrak{g}$,) 98 ($\mathfrak{H}$,) and 98 ($\mathfrak{h}$) bins for a total of 3.9408×10^{11} bins.

For simplicity, we implemented two-body dynamics for this example:

$$\mathbf{X}(t; \mathbf{X}^0) = \left(\mathfrak{L}^0, \mathfrak{l}^0 + \mu^2 t/(\mathfrak{L}^0)^2, \mathfrak{G}^0, \mathfrak{g}^0, \mathfrak{H}^0, \mathfrak{h}^0\right), \tag{29}$$

Since two-body dynamics is deterministic, results in Theorem 2.3 apply. We may implement more complex dynamical models as long as they are deterministic, such as those including atmospheric drag [29], effects due to the oblateness of the Earth [30], solar radiation pressure, etc.

Fig. 1 is a graphical representation of the process explained in Section II-B for two observations of the same object: Object 1. The first observation is taken at the epoch time and the second observation 26 hours later. The top two horizontal sets of contour plots represent the value of the generalized admissible regions that have been dynamically evolved to a common epoch and projected onto the 3 2-dimensional subsets of the Poincaré space. Even though the generalized admissible region was defined to be a uniform pdf in the topocentric spherical coordinates space, the pdf is non-uniform in the Poincaré space due to the non-linear transformation between the two spaces. The time propagation has also *shredded* the pdf for the second observation in the $\mathfrak{L}$-$\mathfrak{l}$ plane [28]. The bottom set of plots is the combined distribution $h[\mathbf{X}(\tau)]$. The yellow asterisk is the true state of the observed object. Note that although these distributions are plotted on 2-dimensional subspaces, the correlation was conducted in the full 6-dimensional Poicaré space. When associating two observations of the same object, we see that $h > 0$ for a very small region of the state space; for this particular example, $h > 0$ for 4 bins. Furthermore, the true state is included in the region in state space where $h > 0$. Therefore, the state estimate is good.

Fig. 2 is a graphical representation of the association of two observations of separate objects. The first observation of Object 1 is again taken at the epoch time and the second observation of Object 2 19 hours later. We find that regions exist where $h > 0$ even though we expect $h = 0$ for all bins. These fictitious *multi-rev* solutions are due to the ambiguity of the number of orbit revolutions between observations [18]. Again, the generalized admissible regions, and subsequently their combined pdfs, indicate the bounds of one's knowledge regarding an object's state: with just these two observations, their association cannot be confidently inferred. If we are to add an observation of Object 2 taken 29 hours after epoch, we eliminate all multi-rev solutions and the combined pdf $h = 0$ as expected.

IV. Conclusion

In this paper, a new approach to the estimation of Earth-orbiting objects was proposed, where observations are expressed as pdfs that represent not only their errors but also the limited knowledge in the unobserved variables. The pdfs, referred to as generalized admissible regions, are bounded by a set of physical constraints. Bayes' rule is employed to rationally combine multiple observations and other probabilistic data. For deterministic systems, the Fokker-Planck equation, which dictate the time propagation of pdfs, has a special solution allowing for an analytical description of the pdf for all time. Through an example for ground-based error-free optical observations, our method was shown to be effective.

Appendix A
Definition of Poincaré Orbital Elements

The Poincaré orbital elements are defined here in terms of a transformation from the topocentric spherical coordinates. The transformation is performed in several steps. First, from topocentric spherical coordinates to geocentric Cartesian coordinates:

$$T_1 : \langle \rho, \dot{\rho}, \mathfrak{X} \rangle \to \langle x, y, z, \dot{x}, \dot{y}, \dot{z} \rangle,$$

then to orbital elements [9]:

$$T_2 : \langle x, y, z, \dot{x}, \dot{y}, \dot{z} \rangle \to \langle a, e, i, \Omega, \omega, M \rangle,$$

where a is the semi-major axis, e is the eccentricity, $i \in [0, \pi]$ is the inclination, $\Omega \in [-\pi, \pi]$ is the longitude of the ascending node, $\omega \in [-\pi, \pi]$ is the argument of periapsis, and $M \in [-\pi, \pi]$ is the mean anomaly. Finally, we transform the orbital elements to Poincaré variables:

$$T_3 : \langle a, e, i, \Omega, \omega, M \rangle \to \langle \mathfrak{L}, \mathfrak{l}, \mathfrak{G}, \mathfrak{g}, \mathfrak{H}, \mathfrak{h} \rangle,$$

which are defined as:

$$\begin{aligned} \mathfrak{l} &= \Omega + \omega + M & \mathfrak{L} &= \sqrt{\mu a} \\ \mathfrak{g} &= \sqrt{2\mathfrak{L}\left(1 - \sqrt{1 - e^2}\right)} \cos(\omega + \Omega) & \mathfrak{G} &= -\mathfrak{g} \tan(\omega + \Omega) \\ \mathfrak{h} &= \sqrt{2\mathfrak{L}\sqrt{1 - e^2}\,(1 - \cos i)} \cos \Omega & \mathfrak{H} &= -\mathfrak{h} \tan \Omega, \end{aligned}$$

where μ is the standard gravitational parameter.

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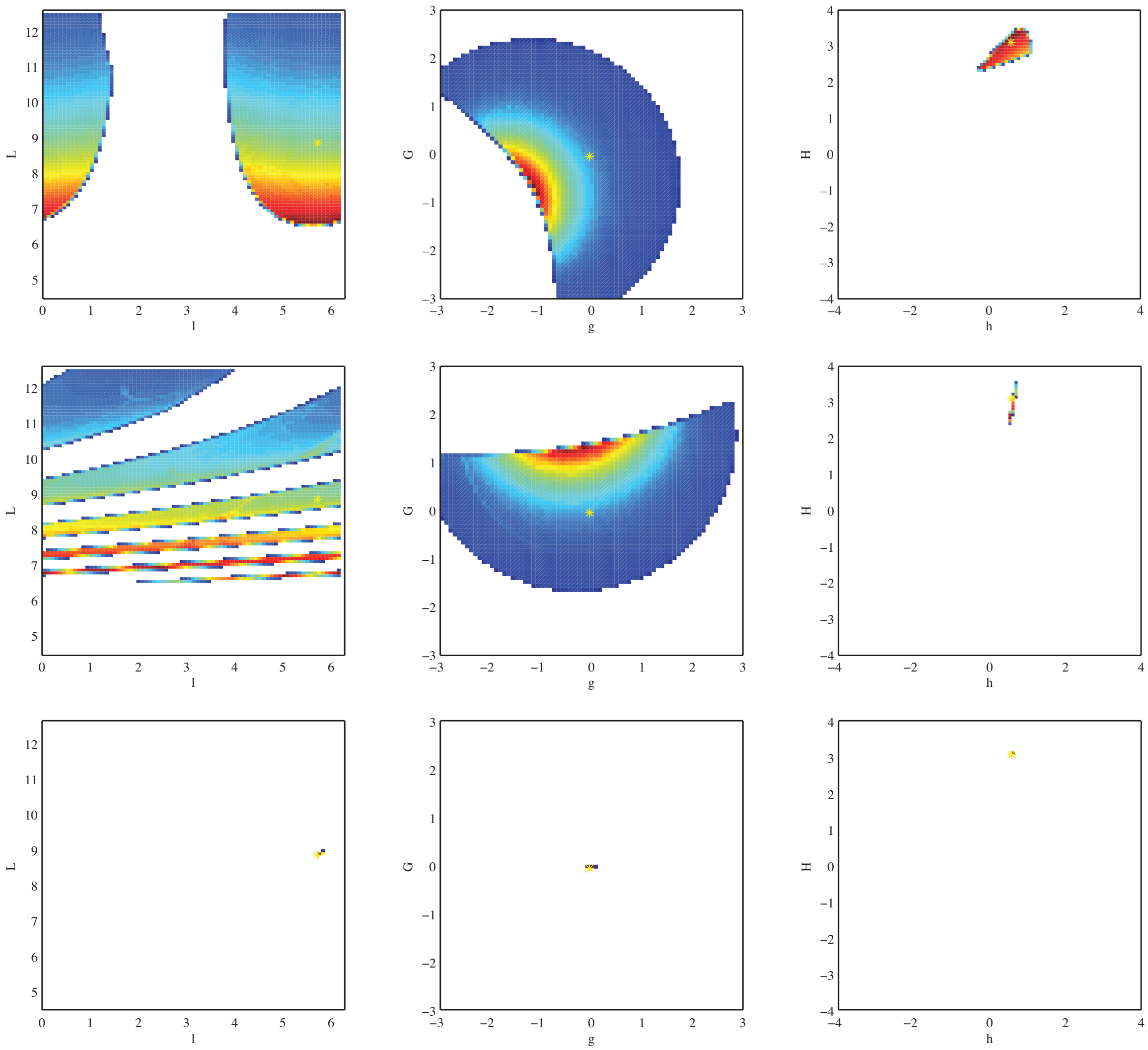

Fig. 1. Projections of generalized admissible regions of two observations of Object 1 (top, middle) and their combined pdf (bottom). Length units in Earth radii, time units in hours, mass units in kilograms.

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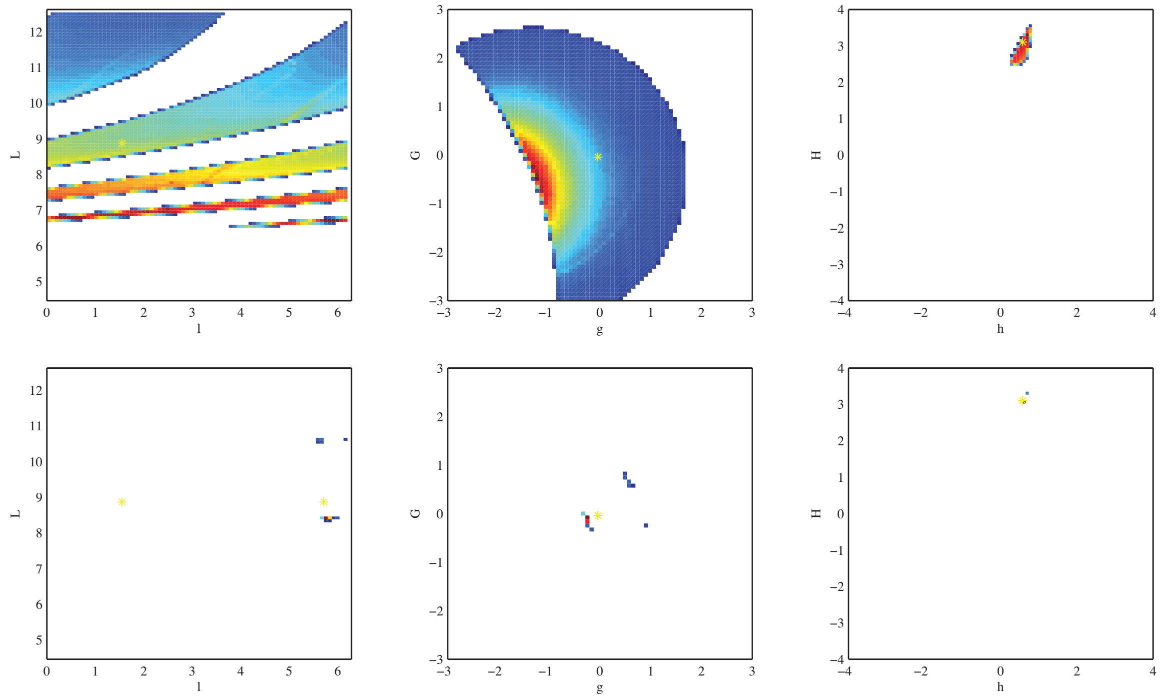

Fig. 2. Projections of generalized admissible regions of two observations of separate objects (top of this figure and Fig. 1) and their combined pdf (bottom). Length units in Earth radii, time units in hours, mass units in kilograms.

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